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Dynamic history-dependent variational-hemivariational inequalities with applications to contact mechanics

Stanislaw Migórski and Justyna Ogorzaly

Abstract. In the paper we deliver a new existence and uniqueness result for a class of abstract nonlinear variational-hemivariational inequalities which are governed by two operators depending on the history of the solution, and include two nondifferentiable functionals, a convex and a nonconvex one. Then, we consider an initial boundary value problem which describes a model of evolution of a viscoelastic body in contact with a foundation. The contact process is assumed to be dynamic, and the friction is described by subdifferential boundary conditions. Both the constitutive law and the contact condition involve memory operators. As an application of the abstract theory, we provide a result on the unique weak solvability of the contact problem.

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1. Introduction

The study of differential equations with constraints has a long history and is closely connected to the study of variational inequalities. The beginning of the research on variational inequalities is due to a contact problem posed by Signorini. The term "variational inequality" was introduced by Fichera and the mathematical theory of variational inequalities started with Stampacchia, who influenced several mathematicians, such as Lions, Hartman, Duvaut, Brezis and others. Existence and uniqueness results for variational inequalities can be found in [2,3,9,20,21].

The notion of a hemivariational inequality, a useful generalization of variational inequality, is concerned with nonconvex and nonsmooth energy functionals and was introduced and studied in the early 1980s by Panagiotopoulos in [36,38,39]. This type of inequality is based on the notion of the generalized directional derivative and the Clarke generalized gradient of a locally Lipschitz function and is closely related to a class of nonlinear inclusions of subdifferential type. During the last four decades, the number of contributions to the area of variational and hemivariational inequalities was enormous, both in the theory and applications, cf. e.g., [9,10,17,20,27,30,36,39,41] and the references therein. A part of this progress was motivated by new models and their formulations arising in Contact Mechanics, cf. e.g., [16,17,35,40,41,45]. Hemivariational inequalities and their systems play nowadays a crucial role in a description of

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many important problems arising in engineering and mechanics, especially in mathematical modeling of various processes involved in contact between deformable bodies.

Recently there has been increasing interest in the theory and applications of variational-hemivariational inequalities that represent a special class of inequalities involving both convex and nonconvex functionals, cf. [14,15,28,32,35]. Recent results in this area deal with variational-hemivariational inequalities with history-dependent operators, cf. e.g., [4,11,19,26,29,31,33,42–44,46,47]. A class of stationary variational-hemivariational inequalities was studied in [15] where results on existence and uniqueness of the solution, continuous dependence of the solution on the data and numerical algorithms for solving such inequality were delivered. These results were applied to a variational-hemivariational inequality arising in the study of quasistatic model of elastic contact. An evolutionary version of the variational-hemivariational inequalities studied in [15] was considered in [31]. There, a quasistatic viscoelastic frictionless contact problem with the memory term was analyzed on unbounded time interval. The paper [43], being the continuation of [31], developed further the theory of variational-hemivariational inequalities and provided the numerical study of quasistatic frictional viscoelastic contact problem. A number of quasistatic contact problems modeled by history-dependent variational inequalities were studied in several papers. For instance, in [44] quasi-variational inequalities are used to deal with contact problems with normal compliance, with normal damped response, and with the Signorini condition. The frictionless contact problem with normal compliance, unilateral constraint and memory effects was investigated in [46]. In contrast to [46], paper [42] examined the frictional contact with normal compliance, memory term, and the Coulomb law of dry friction. All aforementioned papers studied the static and quasistatic problems for history-dependent variational and history-dependent variational-hemivariational inequalities.

The purpose of this paper is to extend a part of these results to a new class of evolutionary problems. First, we prove the unique weak solvability of abstract evolutionary variational-hemivariational inequality in which the derivatives of the unknown variable are involved. In contrast to [25–29,31], no Clarke regularity of locally Lipschitz potential is assumed. We deal with a class of abstract evolution variational-hemivariational inequalities of first order involving history-dependent operators. We study the Cauchy problem for inequality from this class and provide conditions under which the Cauchy problem has a unique solution. The two main features of the variational-hemivariational inequality under investigation are the following. On the one hand, it involves two nondifferentiable potentials, one of them is locally Lipschitz continuous and nonconvex, and the second one is assumed to be convex and lower semicontinuous. On the other hand, the inequality contains two nonlinear operators of history type, and one of them appears in the convex potential. The main result of this paper on existence and uniqueness of solution to variational-hemivariational inequality with history-dependent operators is new and has not been delivered in the literature so far. Our main result is obtained by combining a fixed point argument, already used in several papers, see e.g., [1,12,13], and a recent result for evolution subdifferential inclusions provided in [32].

The class of evolution variational-hemivariational inequalities presented in this paper provides a new mathematical tool and a general framework for a large number of dynamic contact problems, associated with various constitutive laws and frictional or frictionless contact conditions. As an illustration, in the second part of the paper, we consider an initial boundary value problem which describe a model of evolution of a viscoelastic body in contact with a foundation. We assume that the contact process is dynamic and the friction is described by subdifferential boundary conditions. Both the constitutive law and the contact condition involve memory operators. Such kind of problems leads to a new and nonstandard mathematical model. As an application of our abstract result, we provide a theorem on the unique weak solvability of the contact problem.

We also mention that the dynamic Signorini frictionless contact problem for viscoelastic materials with singular memory has been studied by Jarušek [18]. Cocou [6] proved existence of weak solution for a dynamic viscoelastic unilateral contact problem with nonlocal friction and the Kelvin–Voigt law. Results on dynamic contact with velocity dependent friction can be found in Kuttler and Shillor in [22,23]. Note that dynamic contact problems with history-dependent operators have been also studied in two very recent papers [34] and [37]. Theorem 6 of the present paper is an extension to variational-hemivariational inequalities of Theorem 5 from [34] obtained for variational inequalities.

The paper is structured as follows. In Section 2 we recall notation and present some auxiliary material. Section 3 provides the proof of existence and uniqueness result for abstract evolution variational-hemivariational inequalities with history-dependent operators. A dynamic frictional contact problem for viscoelastic materials with long memory is studied in Section 4. We give its variational formulation and show its unique solvability.

2. Preliminaries

In this paper we use standard notation for the Lebesgue and Sobolev spaces of functions defined on a time interval $[0,T]$, $0 < T < +\infty$ with values in a Banach space E with a norm $\|\cdot\|_E$. Recall that the space $L^2(0,T;E)$ of vector-valued functions consists of all measurable functions $u\colon (0,T) \to E$ for which $\int_0^T \|u(t)\|_E^2\,dt$ is finite. The duality pairing between E^* and E is denoted by $\langle\cdot,\cdot\rangle_{E^*\times E}$, where E^* stands for the dual space to E. For a set $U \subset E$, we define $\|U\|_E = \sup\{\|u\|_E \mid u \in U\}$. We denote by $\mathcal{L}(E,F)$ a space of linear and bounded operators from a Banach space E with values in a Banach space F with the usual norm $\|\cdot\|_{\mathcal{L}(E,F)}$. The inner product in a Hilbert space E is denoted by $(\cdot,\cdot)_E$.

In what follows we consider an evolution triple of spaces (V,H,V^*). This means that V is a reflexive separable Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous, and V is dense in H. In this setting the space H is identified with its dual and we have $V \subset H \subset V^*$ with dense and continuous embeddings. We introduce the spaces $\mathcal{V} = L^2(0,T;V)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$, where $\mathcal{V}^* = L^2(0,T;V^*)$ and the derivative is understood in the sense of vector-valued distributions. The space $\mathcal{W}$ endowed with the norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*}$ becomes a separable and reflexive Banach space. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is given by

$$\langle w,v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle w(t),v(t)\rangle_{V^*\times V}\,\mathrm{d}t \quad \text{for} \quad w \in \mathcal{V}^*,\ v \in \mathcal{V}.$$

We have the following continuous embeddings $\mathcal{W} \subset \mathcal{V} \subset L^2(0,T;H) \subset \mathcal{V}^*$. It is well known that $\mathcal{W} \subset C(0,T;H)$ continuously (cf. Proposition 3.4.14 of [8]), where $C(0,T;H)$ stands for the space of continuous functions on $[0,T]$ with values in H.

We recall some facts from the theory monotone operators and convex functions. Let E be a Banach space. An operator $T\colon E \to 2^{E^*}$ is called monotone if $\langle u^*-v^*,u-v\rangle_{E^*\times E} \geq 0$ for all $u^* \in Tu$, $v^* \in Tv$, u, $v \in E$. It is called maximal monotone, if it is monotone and maximal in the sense of inclusion of graphs in the family of monotone operators from E to 2^{E^*}. Operator T is called coercive, if there exists a function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{r\to+\infty}\alpha(r) = +\infty$ such that $\langle u^*,u\rangle \geq \alpha(\|u\|_E)\,\|u\|_E$ for all $u \in E$, $u^* \in Tu$. A single-valued operator $A\colon E \to E^*$ is called pseudomonotone, if it is bounded (it maps bounded sets of E into bounded sets of E^*) and $u_n \to u$ weakly in E with $\limsup\langle Au_n,u_n-u\rangle_{E^*\times E} \leq 0$ imply $\langle Au,u-v\rangle_{E^*\times E} \leq \liminf\langle Au_n,u_n-v\rangle_{E^*\times E}$ for all $v \in E$.

Given an operator $A\colon (0,T)\times E \to E^*$, its Nemitskii (or superposition) operator is the operator $\mathcal{A}\colon L^2(0,T;E) \to L^2(0,T;E^*)$ defined by $(\mathcal{A}v)(t) = A(t,v(t))$ for $v \in L^2(0,T;E)$ and $t \in (0,T)$.

Recall also that a function $\varphi\colon E \to \mathbb{R}\cup\{+\infty\}$ is proper if it is not identically equal to $+\infty$, i.e., the effective domain $\operatorname{dom}\varphi = \{x \in E \mid \varphi(x) < +\infty\} \neq \emptyset$. It is lower semicontinuous (l.s.c.) if $x_n \to x$ in

E implies $\varphi(x) \le \liminf \varphi(x_n)$. It is well known (cf. Proposition 5.2.10 of [7]) that a convex and l.s.c. function $\varphi\colon E \to \mathbb{R}$ defined on a Banach space E, is locally Lipschitz

Definition 1. Let $\varphi\colon E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The subdifferential of $\partial\varphi$ is generally multivalued mapping $\partial\varphi\colon E \to 2^{E^*}$ defined by

$$\partial\varphi(x) = \{\, x^* \in E^* \mid \langle x^*, v - x\rangle_{E^* \times E} \le \varphi(v) - \varphi(x) \text{ for all } v \in E \,\}$$

is called the subdifferential of φ. The elements of the set $\partial\varphi(x)$ are called subgradients of φ in x.

The following fact will be useful in the next sections.

Remark 2. Let E be a Banach space and $\varphi\colon E \to \mathbb{R}$ be a convex and Lipschitz continuous function with constant L_φ. Then $\|\partial\varphi(x)\|_{E^*} \le L_\varphi$ for all $x \in E$.

Finally, we recall the following notions for locally Lipschitz functions.

Definition 3. Let $h\colon E \to \mathbb{R}$ be a locally Lipschitz function on a Banach space E. For $x, v \in E$, the generalized directional derivative of h at x in the direction v, denoted by $h^0(x; v)$ is defined by

$$h^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The generalized (Clarke) gradient (subdifferential) of h at x, denoted by $\partial h(x)$, is a subset of E^* given by

$$\partial h(x) = \{\, \zeta \in E^* \mid h^0(x; v) \ge \langle \zeta, v\rangle_{E^* \times E} \text{ for all } v \in E \,\}.$$

The proofs of the results presented in this section can be found in standard textbooks, e.g., [5,7,8,30,48].

3. Existence and uniqueness result

In this section we provide the existence and uniqueness result for an abstract evolution variational-hemivariational inequality. Its proof is based on a recent result on an evolution inclusion in Banach spaces of [30] and a fixed point argument. We work in the framework of evolution triple of spaces (V, H, V^*). Let X and Y be separable and reflexive Banach spaces.

Consider the operators $A\colon (0, T) \times V \to V^*$, $\mathcal{R}\colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{R}_1\colon \mathcal{V} \to L^2(0, T; Y)$, $M\colon V \to X$, and the functions $J\colon (0, T) \times X \to \mathbb{R}$, $\varphi\colon Y \times X \to \mathbb{R}$ and $f\colon (0, T) \to V^*$.

With these data we consider the following dynamic problem.

Problem 4. *Find $w \in \mathcal{W}$ such that*

$$\begin{aligned}
&\langle w'(t) + A(t, w(t)) + (\mathcal{R}w)(t), v - w(t)\rangle_{V^* \times V} \\
&\quad + J^0(t, Mw(t); Mv - Mw(t)) + \varphi((\mathcal{R}_1 w)(t), Mv) - \varphi((\mathcal{R}_1 w)(t), Mw(t)) \\
&\quad \ge \langle f(t), v - w(t)\rangle_{V^* \times V} \quad \text{for all} \quad v \in V, \text{ a.e. } t \in (0, T), \\
&w(0) = v_0.
\end{aligned}$$

In the study of Problem 4 we will need the following hypotheses on the data.

$$\left.\begin{array}{l} A\colon (0,T)\times V\to V^* \text{ is such that}\\ \text{(a) } A(\cdot,v) \text{ is measurable on } (0,T) \text{ for all } v\in V.\\ \text{(b) } A(t,\cdot) \text{ is pseudomonotone on } V \text{ for a.e. } t\in(0,T).\\ \text{(c) } \|A(t,v)\|_{V^*}\le a_0(t)+a_1\|v\|_V \text{ for all } v\in V,\ \text{a.e. } t\in(0,T)\\ \qquad \text{with } a_0\in L^2(0,T), a_0\ge 0 \text{ and } a_1>0.\\ \text{(d) } \langle A(t,v),v\rangle_{V^*\times V}\ge \alpha_A\|v\|_V^2 \text{ for all } v\in V,\ \text{a.e. } t\in(0,T)\\ \qquad \text{with } \alpha_A>0.\\ \text{(e) } A(t,\cdot) \text{ is strongly monotone for a.e. } t\in(0,T),\ \text{i.e., there}\\ \qquad \text{is } m_A>0 \text{ such that for all } v_1,v_2\in V,\ \text{a.e. } t\in(0,T)\\ \qquad \langle A(t,v_1)-A(t,v_2),v_1-v_2\rangle_{V^*\times V}\ge m_A\|v_1-v_2\|_V^2. \end{array}\right\}\quad (1)$$

$$\left.\begin{array}{l} M\in\mathcal{L}(V,X) \text{ is such that its Nemytskii operator}\\ \qquad \mathcal{M}\colon \mathcal{W}\subset\mathcal{V}\to L^2(0,T;X) \text{ is compact.}\end{array}\right\}\quad (2)$$

$$\left.\begin{array}{l} J\colon (0,T)\times X\to\mathbb{R} \text{ is such that}\\ \text{(a) } J(\cdot,v) \text{ is measurable on } (0,T) \text{ for all } v\in X.\\ \text{(b) } J(t,\cdot) \text{ is locally Lipschitz on } X \text{ for a.e. } t\in(0,T).\\ \text{(c) } \|\partial J(t,v)\|_{X^*}\le c_0(t)+c_1\|v\|_X \text{ for all } v\in X,\\ \qquad \text{a.e. } t\in(0,T) \text{ with } c_0\in L^2(0,T), c_0\ge 0, c_1>0.\\ \text{(d) } \langle v_1^*-v_2^*,v_1-v_2\rangle_{X^*\times X}\ge -m_J\|v_1-v_2\|_X^2 \text{ for all}\\ \qquad v_i^*\in\partial J(t,v_i),\ v_i^*\in X^*,\ v_i\in X,\ i=1,2,\ \text{a.e. } t\in(0,T)\\ \qquad \text{with } m_J\ge 0.\end{array}\right\}\quad (3)$$

$$\left.\begin{array}{l} \varphi\colon Y\times X\to\mathbb{R} \text{ is such that}\\ \text{(a) } \varphi(\cdot,z) \text{ is continuous on } Y \text{ for all } z\in X.\\ \text{(b) } \varphi(y,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } y\in Y.\\ \text{(c) } \|\partial\varphi(y,z)\|_{X^*}\le c_\varphi(1+\|y\|_Y+\|z\|_X) \text{ for all } y\in Y, z\in X,\\ \qquad \text{with } c_\varphi>0.\\ \text{(d) } \varphi(y_1,z_2)-\varphi(y_1,z_1)+\varphi(y_2,z_1)-\varphi(y_2,z_2)\le\\ \qquad \le\beta_\varphi\|y_1-y_2\|_Y\|z_1-z_2\|_X \ \text{ for all } \ y_1,y_2\in Y,\ z_1,z_2\in X\\ \qquad \text{with } \beta_\varphi>0.\end{array}\right\}\quad (4)$$

$$\left.\begin{array}{l} \text{One of the following conditions is satisfied.}\\ \text{(a) } \alpha_A>2\sqrt{2}\,(c_1+c_\varphi)\,\|M\|^2,\ \text{where } \|M\|=\|M\|_{\mathcal{L}(V,X)}.\\ \text{(b) } J^0(t,z;-z)\le d_0\,(1+\|z\|_X) \text{ for all } z\in X, \text{a.e. } t\in(0,T)\\ \qquad \text{with } d_0\ge 0 \ \text{ and } \|\partial\varphi(y,z)\|_{X^*}\le c_\varphi \ \text{ for all } y\in Y,\\ \qquad z\in Z \ \text{ with } c_\varphi\ge 0.\end{array}\right\}\quad (5)$$

$$m_A>m_J\,\|M\|^2. \quad (6)$$

$$f\in L^2(0,T;V^*),\ v_0\in V. \quad (7)$$

$$\left.\begin{array}{l}\mathcal{R}\colon \mathcal{V} \to L^2(0,T;V^*) \text{ and } \mathcal{R}_1\colon \mathcal{V} \to L^2(0,T;Y) \text{ are such that} \\ \text{(a) } \|(\mathcal{R}v_1)(t) - (\mathcal{R}v_2)(t)\|_{V^*} \le c_R \int\limits_0^t \|v_1(s) - v_2(s)\|_V \, \mathrm{d}s \\ \quad \text{for all } v_1, v_2 \in \mathcal{V}, \text{ a.e. } t \in (0,T) \text{ with } c_R > 0. \\ \text{(b) } \|(\mathcal{R}_1 v_1)(t) - (\mathcal{R}_1 v_2)(t)\|_{Y} \le c_{R_1} \int\limits_0^t \|v_1(s) - v_2(s)\|_V \, \mathrm{d}s \\ \quad \text{for all } v_1, v_2 \in \mathcal{V}, \text{ a.e. } t \in (0,T) \text{ with } c_{R_1} > 0. \end{array}\right\} \tag{8}$$

We comment that the function J is locally Lipschitz with respect to the second argument and it is, in general, nonconvex while φ is supposed to be convex and l.s.c. with respect to its second argument. For this reason, the inequality in Problem 4 is called variational-hemivariational inequality.

Remark 5. Hypothesis (3)(d) is called the relaxed monotonicity condition for a locally Lipschitz function $J(t,\cdot)$. It was used in the literature (cf. Section 3.3 of [30]) to guarantee the uniqueness of the solution to hemivariational inequalities. This hypothesis is equivalent to the following condition

$$J^0(t, v_1; v_2 - v_1) + J^0(t, v_2; v_1 - v_2) \le m_J \|v_1 - v_2\|_X^2 \tag{9}$$

for all v_1, $v_2 \in X$, a.e. $t \in (0,T)$. Examples of nonconvex functions which satisfy the relaxed monotonicity condition can be found in [29,30]. It can be proved that for a convex function condition (3)(d), or equivalently (9), holds with $m_J = 0$.

We have the following existence and uniqueness result.

Theorem 6. *Under hypotheses (1)–(8), Problem 4 has a unique solution $w \in \mathcal{W}$.*

Proof. The proof of the theorem will be established in several steps. It is based on a recent result on existence of solution to subdifferential inclusions in [32] and a fixed point argument.

Step 1. First, we fix $\xi \in L^2(0,T;V^*)$ and $\eta \in L^2(0,T;Y)$. Consider the following auxiliary problem.

Problem 7. *Find $w_{\xi\eta} \in \mathcal{W}$ such that*

$$\begin{aligned} &\langle w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)) + \xi(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} \\ &\quad + J^0(t, Mw_{\xi\eta}(t); Mv - Mw_{\xi\eta}(t)) + \varphi(\eta(t), Mv) - \varphi(\eta(t), Mw_{\xi\eta}(t)) \\ &\ge \langle f(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} \quad \text{for all } \ v \in V, \text{ a.e. } t \in (0,T), \\ &\quad w_{\xi\eta}(0) = v_0. \end{aligned}$$

Our goal in this step is to prove that Problem 7 has a unique solution. To this end, define the function $\Psi_\eta\colon (0,T) \times X \to \mathbb{R}$ by

$$\Psi_\eta(t,z) = J(t,z) + \varphi(\eta(t), z) \ \text{ for all } z \in X, \text{ a.e. } t \in (0,T). \tag{10}$$

We claim that under assumptions (3) and (4), the function $\Psi_\eta\colon (0,T) \times X \to \mathbb{R}$ defined by (10) has the following properties.

$$\left.\begin{array}{ll} \text{(a)} & \Psi_\eta(\cdot, z) \text{ is measurable on } (0,T) \text{ for all } z \in X. \\ \text{(b)} & \Psi_\eta(t,\cdot) \text{ is locally Lipschitz on } X \text{ for a.e. } t \in (0,T). \\ \text{(c)} & \|\partial\Psi_\eta(t,z)\|_{X^*} \le \alpha(t) + (c_1 + c_\varphi)\|z\|_X \text{ for all } z \in X \\ & \text{and a.e. } t \in (0,T) \text{ with } \alpha \in L^2(0,T), \ \alpha > 0. \\ \text{(d)} & \langle \partial\Psi_\eta(t,z_1) - \partial\Psi_\eta(t,z_2), z_1 - z_2\rangle_{X^*\times X} \ge -m_J\|z_1 - z_2\|_X^2 \\ & \text{for all } z_1, z_2 \in X \text{ and a.e. } t \in (0,T). \end{array}\right\} \tag{11}$$

Indeed, since $(0,T) \ni t \to \eta(t) \in Y$ is measurable, conditions (3)(a) and (4)(a) imply that the function Ψ_η is also measurable, i.e., (11)(a) is satisfied. Since $\varphi(y,\cdot)$ is convex, lower semicontinuous and finite for $y \in Y$, we know that $\varphi(y,\cdot)$ is locally Lipschitz for $y \in Y$. Hence and by condition (3)(b), we conclude that the function $\Psi_\eta(t,\cdot)$ is locally Lipschitz on X for a.e. $t \in (0,T)$, i.e., (11)(b) holds.

From the fact that $J(t,\cdot)$ and $\varphi(t,\cdot)$ are locally Lipschitz for a.e. $t \in (0,T)$, by Proposition 5.6.23 of [7], we have

$$\partial\Psi_\eta(t,z) \subseteq \partial J(t,z) + \partial\varphi(\eta(t),z) \quad \text{for all } z \in X \text{ and a.e. } t \in (0,T). \tag{12}$$

Hence

$$\begin{aligned}\|\partial\Psi_\eta(t,z)\|_{X^*} &\le \|\partial J(t,z)\|_{X^*} + \|\partial\varphi(\eta(t),z)\|_{X^*}\\ &\le (c_0(t) + c_1\|z\|_X) + c_\varphi(1 + \|\eta(t)\|_Y + \|z\|_X) = \alpha(t) + (c_1 + c_\varphi)\|z\|_X\end{aligned}$$

for all $z \in X$, a.e. $t \in (0,T)$, where $\alpha \in L^2(0,T)$, $\alpha > 0$. So, condition (11)(c) is satisfied.

Finally, since $\varphi(y,\cdot)$ is convex and lower semicontinuous by (4)(b), from Theorem 6.3.19 in [7], we know that $\partial\varphi(y,\cdot)$ is maximal monotone for all $y \in Y$. Using the monotonicity of $\partial\varphi(y,\cdot)$ for $y \in Y$ and condition (3)(d), we get

$$\begin{aligned}&\langle\partial\Psi_\eta(t,z_1) - \partial\Psi_\eta(t,z_2), z_1 - z_2\rangle_{X^*\times X} = \langle\partial J(t,z_1) - \partial J(t,z_2), z_1 - z_2\rangle_{X^*\times X}\\ &\quad + \langle\partial\varphi(\eta(t),z_1) - \partial\varphi(\eta(t),z_2), z_1 - z_2\rangle_{X^*\times X} \ge -m_J\|z_1 - z_2\|_X^2\end{aligned}$$

for all z_1, $z_2 \in X$, a.e. $t \in (0,T)$. Hence condition (11)(d) holds, which completes the proof of (11).

Subsequently, we associate with Problem 7 the following evolutionary inclusion.

$$\left.\begin{aligned}&\text{Find } w_{\xi\eta} \in \mathcal{W} \text{ such that}\\ &w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)) + M^*\partial\Psi_\eta(t, Mw_{\xi\eta}(t)) \ni f_\xi(t) \;\; \text{a.e. } \; t \in (0,T)\\ &w_{\xi\eta}(0) = v_0,\end{aligned}\right\} \tag{13}$$

where $f_\xi \in L^2(0,T;V^*)$ is given by $f_\xi(t) = f(t) - \xi(t)$ for a.e. $t \in (0,T)$ and Ψ_η is defined by (10). Having in mind hypotheses (1)–(7) and properties (11), we are now in a position to apply Theorem 2.6 of [32] to deduce that problem (13) has a unique solution $w_{\xi\eta} \in \mathcal{W}$.

Next, from (12) and (13), we infer that $w_{\xi\eta} \in \mathcal{W}$ is also a solution to the following problem.

$$\left.\begin{aligned}&\text{Find } w_{\xi\eta} \in \mathcal{W} \text{ such that}\\ &w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)) + M^*\partial J(t, Mw_{\xi\eta}(t)) + M^*\partial\varphi(\eta(t), Mw_{\xi\eta}(t)) \ni f_\xi(t)\\ &\hspace{20em}\text{a.e. } \; t \in (0,T)\\ &w_{\xi\eta}(0) = v_0.\end{aligned}\right\} \tag{14}$$

We claim that every solution to inclusion (14) is also a solution to Problem 7. To prove the claim, let $w_{\xi\eta} \in \mathcal{W}$ be the solution to problem (14). This means that there exist $\rho_{\xi\eta}$, $\delta_{\xi\eta} \in L^2(0,T;X^*)$ such that

$$\left.\begin{aligned}&w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)) + M^*\rho_{\xi\eta}(t) + M^*\delta_{\xi\eta}(t) = f_\xi(t) \;\; \text{a.e. } \; t \in (0,T)\\ &\rho_{\xi\eta}(t) \in \partial J(t, Mw_{\xi\eta}(t)) \;\; \text{a.e. } \; t \in (0,T)\\ &\delta_{\xi\eta}(t) \in \partial\varphi(\eta(t), Mw_{\xi\eta}(t)) \;\; \text{a.e. } \; t \in (0,T)\\ &w_{\xi\eta}(0) = v_0.\end{aligned}\right\} \tag{15}$$

By Definitions 1 and 3 of the convex and Clarke subdifferentials, we have

$$\begin{aligned}&\langle\rho_{\xi\eta}(t), z\rangle_{X^*\times X} \le J^0(t, Mw_{\xi\eta}(t); z)\\ &\langle\delta_{\xi\eta}(t), z - Mw_{\xi\eta}(t)\rangle_{X^*\times X} \le \varphi(\eta(t), z) - \varphi(\eta(t), Mw_{\xi\eta}(t))\end{aligned}$$

for all $z \in X$, a.e. $t \in (0,T)$. Let $v \in V$. Multiplying the first equation in (15) by $v - w_{\xi\eta}(t)$, we obtain

$$\langle w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)), v - w_{\xi\eta}(t)\rangle_{V^*\times V} + \langle M^*\rho_{\xi\eta}(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V}$$
$$+ \langle M^*\delta_{\xi\eta}(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} = \langle f_\xi(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V}.$$

Inserting the two inequalities

$$\langle M^*\rho_{\xi\eta}(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} \le J^0(t, Mw_{\xi\eta}(t); Mv - Mw_{\xi\eta}(t)),$$
$$\langle M^*\delta_{\xi\eta}(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} \le \varphi(\eta(t), Mv) - \varphi(\eta(t), Mw_{\xi\eta}(t))$$

for a.e. $t \in (0,T)$, into the above equation, we get

$$\langle w'_{\xi\eta}(t) + A(t, w_{\xi\eta}(t)), v - w_{\xi\eta}(t)\rangle_{V^*\times V}$$
$$+ J^0(t, Mw_{\xi\eta}(t); Mv - Mw_{\xi\eta}(t)) + \varphi(\eta(t), Mv) - \varphi(\eta(t), Mw_{\xi\eta}(t))$$
$$\ge \langle f_\xi(t), v - w_{\xi\eta}(t)\rangle_{V^*\times V} \quad \text{for all} \quad v \in V, \text{ a.e. } t \in (0,T),$$
$$w_{\xi\eta}(0) = v_0.$$

Hence, $w_{\xi\eta} \in \mathcal{W}$ is a solution to Problem 7. This completes the proof of the claim.

To complete the proof of Step 1, we show that the solution to Problem 7 is unique. Let $w_1, w_2 \in \mathcal{W}$ be the solutions to Problem 7 (we skip the subscripts ξ, η for this part of the proof). We write two inequalities: for w_1 and take $w_2(t)$ as the test function, and for w_2 and take $w_1(t)$ as the test function. We have

$$\langle w'_1(t) + A(t, w_1(t)), w_2(t) - w_1(t)\rangle_{V^*\times V}$$
$$+ J^0(t, Mw_1(t); Mw_2(t) - Mw_1(t)) + \varphi(\eta(t), Mw_2(t)) - \varphi(\eta(t), Mw_1(t))$$
$$\ge \langle f_\xi(t), w_2(t) - w_1(t)\rangle_{V^*\times V} \quad \text{a.e. } t \in (0,T)$$

and

$$\langle w'_2(t) + A(t, w_2(t)), w_1(t) - w_2(t)\rangle_{V^*\times V}$$
$$+ J^0(t, Mw_2(t); Mw_1(t) - Mw_2(t)) + \varphi(\eta(t), Mw_1(t)) - \varphi(\eta(t), Mw_2(t))$$
$$\ge \langle f_\xi(t), w_1(t) - w_2(t)\rangle_{V^*\times V} \quad \text{a.e. } t \in (0,T)$$

and $w_1(0) = w_2(0) = v_0$. Adding these inequalities, we deduce

$$\langle w'_1(t) - w'_2(t), w_1(t) - w_2(t)\rangle_{V^*\times V}$$
$$+ \langle A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t)\rangle_{V^*\times V}$$
$$\le J^0(t, Mw_1(t); Mw_2(t) - Mw_1(t)) + J^0(t, Mw_2(t); Mw_1(t) - Mw_2(t))$$

for a.e $t \in (0,T)$. Integrating the above inequality on the time interval $(0,t)$, using the integration by parts formula, cf. e.g., Proposition 3.4.14 in [8], condition (1)(e) and Remark 5, it follows that

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_H^2 - \frac{1}{2}\|w_1(0) - w_2(0)\|_H^2 + m_A \int_0^t \|w_1(s) - w_2(s)\|_V^2 \, \mathrm{d}s$$
$$\le m_J\|M\|^2 \int_0^t \|w_1(s) - w_2(s)\|_V^2 \, \mathrm{d}s \quad \text{for all} \quad t \in [0,T].$$

Hence, from condition $w_1(0) = w_2(0) = v_0$, and assumption (6), we obtain

$$\|w_1(t) - w_2(t)\|_H^2 = 0 \quad \text{for all} \quad t \in [0,T].$$

This implies that $w_1(t) = w_2(t)$ for all $t \in [0, T]$, i.e., $w_1 = w_2$. In conclusion, we deduce that solution to Problem 7 is unique. This completes the proof of Step 1.

Step 2. In this part of the proof, we define the operator $\Lambda\colon L^2(0,T;V^* \times Y) \to L^2(0,T;V^* \times Y)$ by

$$\Lambda(\xi,\eta) = (\mathcal{R}w_{\xi\eta}, \mathcal{R}_1 w_{\xi\eta}) \quad \text{for all} \quad (\xi,\eta) \in L^2(0,T;V^* \times Y),$$

where $w_{\xi\eta} \in \mathcal{W}$ denotes the unique solution to Problem 7 corresponding to (ξ, η).

We show that operator Λ has a unique fixed point. To this end, we apply Lemma 7 of [24]. We will prove that for all (ξ_1, η_1), $(\xi_2, \eta_2) \in L^2(0,T;V^* \times Y)$ and a.e. $t \in (0,T)$, we have

$$\|\Lambda(\xi_1,\eta_1)(t) - \Lambda(\xi_2,\eta_2)(t)\|^2_{V^*\times Y} \le c \int_0^t \|(\xi_1,\eta_1)(s) - (\xi_2,\eta_2)(s)\|^2_{V^*\times Y}\, \mathrm{d}s \tag{16}$$

with $c > 0$.

Let (ξ_1, η_1), $(\xi_2, \eta_2) \in L^2(0,T;V^* \times Y)$ and $w_1 = w_{\xi_1\eta_1}$, $w_2 = w_{\xi_2\eta_2}$ be the unique solutions to Problem 7 corresponding to (ξ_1, η_1) and (ξ_2, η_2), respectively. Thus

$$\begin{aligned}
&\langle w_1'(t) + A(t, w_1(t)), w_2(t) - w_1(t)\rangle_{V^*\times V} \\
&\quad + J^0(t, Mw_1(t); Mw_2(t) - Mw_1(t)) + \varphi(\eta_1(t), Mw_2(t)) - \varphi(\eta_1(t), Mw_1(t)) \\
&\quad \ge \langle f(t) - \xi_1(t), w_2(t) - w_1(t)\rangle_{V^*\times V} \quad \text{a.e. } t \in (0,T)
\end{aligned}$$

and

$$\begin{aligned}
&\langle w_2'(t) + A(t, w_2(t)), w_1(t) - w_2(t)\rangle_{V^*\times V} \\
&\quad + J^0(t, Mw_2(t); Mw_1(t) - Mw_2(t)) + \varphi(\eta_2(t), Mw_1(t)) - \varphi(\eta_2(t), Mw_2(t)) \\
&\quad \ge \langle f(t) - \xi_2(t), w_1(t) - w_2(t)\rangle_{V^*\times V} \quad \text{a.e. } t \in (0,T)
\end{aligned}$$

and $w_1(0) = w_2(0) = v_0$. Adding these two inequalities, we obtain

$$\begin{aligned}
&\langle w_1'(t) - w_2'(t), w_2(t) - w_1(t)\rangle_{V^*\times V} + \langle A(t, w_1(t)) - A(t, w_2(t)), w_2(t) - w_1(t)\rangle_{V^*\times V} \\
&\quad + J^0(t, Mw_1(t); Mw_2(t) - Mw_1(t)) + J^0(t, Mw_2(t); Mw_1(t) - Mw_2(t)) \\
&\quad + \varphi(\eta_1(t), Mw_2(t)) - \varphi(\eta_1(t), Mw_1(t)) + \varphi(\eta_2(t), Mw_1(t)) - \varphi(\eta_2(t), Mw_2(t)) \\
&\quad \ge \langle \xi_1(t), w_2(t) - w_1(t)\rangle_{V^*\times V} - \langle \xi_2(t), w_1(t) - w_2(t)\rangle_{V^*\times V}
\end{aligned}$$

for a.e. $t \in (0,T)$.

Similarly as in the first part of the proof, we integrate the above inequality on $(0,t)$, use the integration by parts formula, and hypotheses (1)(e), (3)(d) and (4)(d). We get

$$\begin{aligned}
&\frac{1}{2}\|w_1(t) - w_2(t)\|^2_H - \frac{1}{2}\|w_1(0) - w_2(0)\|^2_H + m_A \int_0^t \|w_1(s) - w_2(s)\|^2_V\, \mathrm{d}s \\
&\quad \le m_J \|M\|^2 \int_0^t \|w_1(s) - w_2(s)\|^2_V\, \mathrm{d}s \\
&\quad + \beta_\varphi \|M\| \int_0^t \|\eta_1(s) - \eta_2(s)\|_Y \|w_1(s) - w_2(s)\|_V\, \mathrm{d}s \\
&\quad + \int_0^t \|\xi_1(s) - \xi_2(s)\|_{V^*} \|w_1(s) - w_2(s)\|_V\, \mathrm{d}s
\end{aligned}$$

for all $t \in [0, T]$. Using hypothesis (6) and the Hölder inequality, we have

$$\widetilde{c}\,\|w_1 - w_2\|^2_{L^2(0,t;V)} \le \beta_\varphi \|M\|\,\|\eta_1 - \eta_2\|_{L^2(0,t;Y)}\|w_1 - w_2\|_{L^2(0,t;V)}$$
$$+\|\xi_1 - \xi_2\|_{L^2(0,t;V^*)}\|w_1 - w_2\|_{L^2(0,t;V)}$$

for all $t \in [0, T]$ with $\widetilde{c} = m_A - m_J\|M\|^2 > 0$. Thus

$$\|w_1 - w_2\|_{L^2(0,t;V)} \le c\left(\|\eta_1 - \eta_2\|_{L^2(0,t;Y)} + \|\xi_1 - \xi_2\|_{L^2(0,t;V^*)}\right) \tag{17}$$

for all $t \in [0, T]$, where c is a positive constant which value may change from line to line.

On the other hand, by the definition of operator Λ, hypothesis (8), condition (17) and the Jensen inequality, one can verify that

$$\begin{aligned}
&\|\Lambda(\xi_1, \eta_1)(t) - \Lambda(\xi_2, \eta_2)(t)\|^2_{V^*\times Y}\\
&= \|(\mathcal{R}w_1)(t) - (\mathcal{R}w_2)(t)\|^2_{V^*} + \|(\mathcal{R}_1 w_1)(t) - (\mathcal{R}_1 w_2)(t)\|^2_Y\\
&\le \left(c_R \int_0^t \|w_1(s) - w_2(s)\|_V \,\mathrm{d}s\right)^2 + \left(c_{R_1} \int_0^t \|w_1(s) - w_2(s)\|_V \,\mathrm{d}s\right)^2\\
&\le c\,\|w_1 - w_2\|^2_{L^2(0,t;V)} \le c\left(\|\eta_1 - \eta_2\|^2_{L^2(0,t;Y)} + \|\xi_1 - \xi_2\|^2_{L^2(0,t;V^*)}\right)\\
&\le c \int_0^t \|(\xi_1, \eta_1)(s) - (\xi_2, \eta_2)(s)\|^2_{V^*\times Y} \,\mathrm{d}s
\end{aligned}$$

for a.e. $t \in (0, T)$. This proves condition (16) and subsequently, by Lemma 7 of [24], we deduce that there exists a unique fixed point (ξ^*, η^*) of Λ, i.e.,

$$(\xi^*, \eta^*) \in L^2(0, T; V^* \times Y) \text{ is such that } \Lambda(\xi^*, \eta^*) = (\xi^*, \eta^*).$$

The proof of Step 2 is complete.

Step 3. Let $(\xi^*, \eta^*) \in L^2(0, T; V^* \times Y)$ be the unique fixed point of operator Λ. Let $w_{\xi^*\eta^*} \in \mathcal{W}$ be the unique solution to Problem 7 corresponding to (ξ^*, η^*). From the definition of operator Λ, we have

$$\xi^* = \mathcal{R}(w_{\xi^*\eta^*}) \quad\text{and}\quad \eta^* = \mathcal{R}_1(w_{\xi^*\eta^*}).$$

Using these relations in Problem 7, we easily deduce that $w_{\xi^*\eta^*}$ is the unique solution to Problem 4. This completes the proof of the theorem. □

4. A dynamic frictional contact problem

Many important dynamic contact problems dealing with elastic, viscoelastic or viscoplastic materials can be cast in a variational-hemivariational inequality form as in Problem 4 in which the unknown is the velocity field. In this section we provide a description of a dynamic viscoelastic contact problem to which our abstract result of Section 3 can be applied. We show that the variational formulation of the contact problem leads to evolutionary variational-hemivariational inequality for which we prove a result on existence and uniqueness of weak solution.

We start with the notation needed to describe the contact problem and with its physical setting. Then, we provide the hypotheses under which we study the contact problem. We denote by $\mathbb{S}^d$ the space of $d \times d$ symmetric matrices, and we always adopt the summation convention over repeated indices. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i\, v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \text{for all } \ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\,\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \text{for all } \ \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

The physical setting of the contact problem is as follows. A deformable viscoelastic body occupies a set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in applications. The volume forces and surface tractions depend on time and act on the body. We are interested in the dynamic process of the mechanical state of the body on the time interval $[0, T]$ with $0 < T < +\infty$. The boundary $\Gamma = \partial\Omega$ is supposed to be Lipschitz continuous and to be composed of three parts Γ_D, Γ_N and Γ_C which are mutually disjoint, and the measure of Γ_D, denoted by $|\Gamma_D|$, is positive. Then, the unit outward normal vector $\boldsymbol{\nu}$ exists a.e. on Γ. We assume that the body is clamped on part Γ_D, so the displacement field vanishes there. Volume forces of density $\boldsymbol{f}_0$ act in Ω and surface tractions of density $\boldsymbol{f}_N$ are applied on Γ_N. The body may come in contact with an obstacle over the potential contact surface Γ_C. In what follows we put $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $\Sigma_D = \Gamma_D \times (0, T)$, $\Sigma_N = \Gamma_N \times (0, T)$ and $\Sigma_C = \Gamma_C \times (0, T)$. We often do not indicate explicitly the dependence of functions on the spatial variable $\boldsymbol{x} \in \Omega$.

Moreover, for a vector $\boldsymbol{\xi} \in \mathbb{R}^d$, the normal and tangential components of $\boldsymbol{\xi}$ on the boundary are denoted by $\xi_\nu = \boldsymbol{\xi} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\xi}_\tau = \boldsymbol{\xi} - \xi_\nu \boldsymbol{\nu}$, respectively. The normal and tangential components of the matrix $\boldsymbol{\sigma} \in \mathbb{S}^d$ are defined on boundary by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

We denote by $\boldsymbol{u} \colon Q \to \mathbb{R}^d$ the displacement vector, by $\boldsymbol{\sigma} \colon Q \to \mathbb{S}^d$ the stress tensor and by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the linearized (small) strain tensor, where i, $j = 1, \ldots, d$. Recall that the components of the linearized strain tensor are given by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = 1/2(u_{i,j} + u_{j,i})$, where $u_{i,j} = \partial u_i / \partial x_j$.

The classical formulation of the problem reads as follows.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} \colon Q \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} \colon Q \to \mathbb{S}^d$ *such that*

$$\boldsymbol{u}''(t) - \operatorname{Div} \boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \quad \text{in } Q, \tag{18}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathcal{K}(t - s, \boldsymbol{\varepsilon}(\boldsymbol{u}'(s)))\, ds \quad \text{in } Q, \tag{19}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_D, \tag{20}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on } \Sigma_N, \tag{21}$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, u'_\nu(t)) \quad \text{on } \Sigma_C, \tag{22}$$

$$-\boldsymbol{\sigma}_\tau(t) \in h(u_\nu(t))\, \partial\psi(\boldsymbol{u}'_\tau(t)) \quad \text{on } \Sigma_C, \tag{23}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{24}$$

We note that (18) is the equation of motion in which "Div" denotes the divergence operator for tensor valued functions, $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$, and, for simplicity, we assume that the density of mass is equal to one. Equation (19) represents the viscoelastic constitutive law with long memory, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{K}$ are nonlinear time-dependent viscosity, elasticity and relaxation operators, respectively. Conditions (20) and (21) are the displacement and the traction boundary conditions. The multivalued conditions (22) and (23) represent the contact and friction conditions, respectively, in which j_ν, h and ψ are given functions. The function j_ν is locally Lipschitz in the second variable, and ∂j_ν denotes its Clarke subdifferential, while the function ψ is convex in the second variable and $\partial\psi$ stands for its convex subdifferential. Remark that the explicit dependence of the operators $\mathcal{A}$ and $\mathcal{B}$ in (19) and the function j_ν in (22) on the time variable allows to model situation when the frictional contact conditions depend on the temperature, which plays the role of a parameter, i.e., its evolution in time is prescribed. The example of contact condition (22) is the so-called normal damped response condition of the form

$$-\sigma_\nu(t) = k_\nu(\boldsymbol{x}, t) p_\nu(u'_\nu(t)) \quad \text{on} \quad \Sigma_C,$$

where $k_\nu \in L^\infty(\Sigma_C)$ and $p_\nu \colon \mathbb{R} \to \mathbb{R}$ is continuous. In this case, potential $j_\nu(\boldsymbol{x}, t, r) = k_\nu(\boldsymbol{x}, t) \int_0^r p_\nu(s)\, ds$ and $\partial j_\nu(\boldsymbol{x}, t, r) = k_\nu(\boldsymbol{x}, t) p_\nu(r)$ for all $r \in \mathbb{R}$, a.e. $(\boldsymbol{x}, t) \in \Sigma_C$. Since p_ν is not supposed to be increasing, $j_\nu(\boldsymbol{x}, t, \cdot)$ is not necessary a convex function. Various examples of the nonmonotone normal damped response condition are presented in [16,27,30,41,45]. The friction condition (23) incorporates several conditions met in the literature. One of the simplest choices is $\psi(\boldsymbol{x}, \boldsymbol{\xi}) = k(\boldsymbol{x}) \|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_C$, where k is a nonnegative function. This leads to the Coulomb law of dry friction of the form

$$\|\boldsymbol{\sigma}_\tau(t)\| \le F_b, \quad -\boldsymbol{\sigma}_\tau(t) = F_b \, \frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|} \ \text{ if } \ \boldsymbol{u}'_\tau \neq 0,$$

where $F_b = h(u_\nu(t)) k(\boldsymbol{x})$ represents the friction bound. We refer to Section 6.3 of [30] for a detailed discussion on the friction laws of the form (23). Finally, conditions (24) represent the initial conditions where $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ denote the initial displacement and the initial velocity, respectively.

Next, we introduce the spaces needed for the variational formulation. Let

$$V = \{\, \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = 0 \ \text{ on } \Gamma_D \,\}, \tag{25}$$

which is a closed subspace of $H^1(\Omega; \mathbb{R}^d)$ due to the continuity of the trace operator and $H = L^2(\Omega; \mathbb{R}^d)$. Then (V, H, V^*) forms the evolution triple of spaces. It is well known that the trace operator denoted by $\gamma \colon V \to L^2(\Gamma; \mathbb{R}^d)$ is linear and continuous. For the element $\boldsymbol{v} \in V$ we still use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary. We also set $\mathcal{H} = L^2(\Omega; \mathbb{S}^d)$. On V we consider the inner product and the corresponding norm given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_V = \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}}, \quad \|\boldsymbol{v}\| = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ \text{ for } \boldsymbol{u}, \boldsymbol{v} \in V.$$

From the Korn inequality $\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \le c \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}$ for $\boldsymbol{v} \in V$ with $c > 0$, it follows that $\| \cdot \|_{H^1(\Omega;\mathbb{R}^d)}$ and $\| \cdot \|$ are equivalent norms on V.

In the study of problem (18)–(24) we consider the following assumptions on the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$ and the relaxation operator $\mathcal{K}$.

$$\left.\begin{array}{l}
\mathcal{A} \colon Q \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\
\quad \text{(a) } \mathcal{A}(\cdot, \cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } Q \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\quad \text{(b) } \mathcal{A}(\boldsymbol{x}, t, \cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } (\boldsymbol{x}, t) \in Q. \\
\quad \text{(c) } (\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d} \\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x}, t) \in Q \text{ with } m_{\mathcal{A}} > 0. \\
\quad \text{(d) } \|\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \le \overline{a}_0(\boldsymbol{x}, t) + \overline{a}_1 \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\
\qquad\quad \text{a.e. } (\boldsymbol{x}, t) \in Q \text{ with } \overline{a}_0 \in L^2(Q), \overline{a}_0 \ge 0 \text{ and } \overline{a}_1 > 0. \\
\quad \text{(e) } \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } (\boldsymbol{x}, t) \in Q.
\end{array}\right\} \tag{26}$$

$$\left.\begin{array}{l}
\mathcal{B} \colon Q \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\
\quad \text{(a) } \mathcal{B}(\cdot, \cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } Q \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\quad \text{(b) } \|\mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \le L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} \\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x}, t) \in Q \text{ with } L_{\mathcal{B}} > 0. \\
\quad \text{(c) } \mathcal{B}(\cdot, \cdot, \boldsymbol{0}) \in L^2(Q; \mathbb{S}^d).
\end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l}\mathcal{K}\colon Q\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\ \quad\text{(a) } \mathcal{K}(\cdot,\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } Q \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \quad\text{(b) } \|\mathcal{K}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1)-\mathcal{K}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d}\le L_{\mathcal{K}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}\\ \qquad\quad\text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } (\boldsymbol{x},t)\in Q \text{ with } L_{\mathcal{K}}>0.\\ \quad\text{(c) } \mathcal{K}(\cdot,\cdot,\boldsymbol{0})\in L^2(Q;\mathbb{S}^d).\end{array}\right\}\tag{28}$$

The memory function h, the friction potential ψ and the contact potential j_ν satisfy the following hypotheses.

$$\left.\begin{array}{l}h\colon\Gamma_C\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that}\\ \quad\text{(a) } h(\cdot,r) \text{ is measurable on } \Gamma_C \text{ for all } r\in\mathbb{R}.\\ \quad\text{(b) } |h(\boldsymbol{x},r_1)-h(\boldsymbol{x},r_2)|\le L_h|r_1-r_2| \text{ for all}\\ \qquad\quad r_1,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C \text{ with } L_h>0.\\ \quad\text{(c) } 0\le h(\boldsymbol{x},r)\le\overline{h} \text{ for all } r\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C \text{ with } \overline{h}>0.\end{array}\right\}\tag{29}$$

$$\left.\begin{array}{l}\psi\colon\Gamma_C\times\mathbb{R}^d\to\mathbb{R} \text{ is such that}\\ \quad\text{(a) } \psi(\cdot,\boldsymbol{\xi}) \text{ is measurable on } \Gamma_C \text{ for all } \boldsymbol{\xi}\in\mathbb{R} \text{ and}\\ \qquad\quad\text{there exists } \boldsymbol{\xi}_0\in\mathbb{R}^d \text{ such that } \psi(\cdot,\boldsymbol{\xi}_0)\in L^2(\Gamma_C).\\ \quad\text{(b) } \psi(\boldsymbol{x},\cdot) \text{ is convex for a.e. } \boldsymbol{x}\in\Gamma_C.\\ \quad\text{(c) } |\psi(\boldsymbol{x},\boldsymbol{\xi}_1)-\psi(\boldsymbol{x},\boldsymbol{\xi}_2)|\le L_\psi\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\| \text{ for all}\\ \qquad\quad\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^d,\ \text{a.e. } \boldsymbol{x}\in\Gamma_C \text{ with } L_\psi>0.\end{array}\right\}\tag{30}$$

$$\left.\begin{array}{l}j_\nu\colon\Sigma_C\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\ \quad\text{(a) } j_\nu(\cdot,\cdot,r) \text{ is measurable on } \Sigma_C \text{ for all } r\in\mathbb{R} \text{ and there}\\ \qquad\quad\text{exists } \mathbf{e}\in L^2(\Gamma_C) \text{ such that } j_\nu(\cdot,\cdot,\mathbf{e}(\cdot))\in L^1(\Sigma_C).\\ \quad\text{(b) } j_\nu(\boldsymbol{x},t,\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } (\boldsymbol{x},t)\in\Sigma_C.\\ \quad\text{(c) } |\partial j_\nu(\boldsymbol{x},t,r)|\le\overline{c}_{0\nu}(\boldsymbol{x},t)+\overline{c}_{1\nu}|r| \text{ for all } r\in\mathbb{R},\\ \qquad\quad\text{a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ with } \overline{c}_{0\nu}\in L^2(\Sigma_C),\overline{c}_{0\nu},\overline{c}_{1\nu}\ge 0.\\ \quad\text{(d) } (r_1^*-r_2^*)(r_1-r_2)\ge -m_\nu|r_1-r_2|^2 \text{ for all}\\ \qquad\quad r_i^*\in\partial j_\nu(\boldsymbol{x},t,r_i), r_i\in\mathbb{R}, i=1,2,\ \text{a.e. } (\boldsymbol{x},t)\in\Sigma_C\\ \qquad\quad\text{with } m_\nu\ge 0.\\ \quad\text{(e) } j_\nu^0(\boldsymbol{x},t,r;-r)\le\overline{d}_\nu(1+|r|) \text{ for all } r\in\mathbb{R},\\ \qquad\quad\text{a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ with } \overline{d}_\nu\ge 0.\end{array}\right\}\tag{31}$$

We suppose that the densities of the body forces and tractions, and the initial data have the following regularity

$$\boldsymbol{f}_0\in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\quad \boldsymbol{f}_N\in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)),\quad \boldsymbol{u}_0,\,\boldsymbol{v}_0\in V.\tag{32}$$

Finally, we need the following smallness condition

$$m_{\mathcal{A}}>m_\nu\|\gamma\|^2.\tag{33}$$

We now pass to the variational formulation of Problem $\mathcal{P}$. Let $(\boldsymbol{u},\boldsymbol{\sigma})$ be a couple of sufficiently smooth functions which solve (18)–(23). Let $\boldsymbol{v}\in V$ and $t\in(0,T)$. Then, using the Green formula (cf. Theorem 2.25 in [30]) and (18), we have

$$\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v}-\boldsymbol{u}'(t))\,\mathrm{d}x+\int_\Omega \boldsymbol{\sigma}(t):(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))\,\mathrm{d}x$$

$$= \int_{\Gamma} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}'(t))\, \mathrm{d}\Gamma + \int_{\Omega} \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}'(t))\, \mathrm{d}x.$$

Next, we employ boundary conditions (20) and (21) and the decomposition formula $\boldsymbol{\sigma}(t)\,\boldsymbol{\nu}\cdot\boldsymbol{v} = \sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau$ on Σ_C to obtain

$$\begin{aligned}
&\int_{\Omega} \boldsymbol{u}''(t) \cdot (\boldsymbol{v} - \boldsymbol{u}'(t))\, \mathrm{d}x + \int_{\Omega} \boldsymbol{\sigma}(t) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))\, \mathrm{d}x \\
&\quad - \int_{\Gamma_C} \sigma_\nu(t)(v_\nu - u'_\nu(t))\, \mathrm{d}\Gamma - \int_{\Gamma_C} \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}'_\tau(t))\, \mathrm{d}\Gamma \\
&= \int_{\Omega} \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}'(t))\, \mathrm{d}x + \int_{\Gamma_N} \boldsymbol{f}_N(t) \cdot (\boldsymbol{v} - \boldsymbol{u}'(t))\, \mathrm{d}\Gamma. \qquad (34)
\end{aligned}$$

Subsequently, from boundary conditions (22) and (23), by definitions of the convex and Clarke subdifferentials, we infer

$$-\sigma_\nu(t)(v_\nu - u'_\nu(t)) \le j^0_\nu(t, u'_\nu(t); v_\nu - u'_\nu(t)),$$
$$-\boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}'_\tau(t)) \le h(u_\nu(t))(\psi(\boldsymbol{v}_\tau) - \psi(\boldsymbol{u}'_\tau(t)))$$

on Σ_C, for a.e. $t \in (0,T)$. Using these inequalities in (34), we obtain

$$\begin{aligned}
&\langle \boldsymbol{u}''(t), \boldsymbol{v} - \boldsymbol{u}'(t)\rangle_{V^*\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))_{\mathcal{H}} + \int_{\Gamma_C} j^0_\nu(t, u'_\nu(t); v_\nu - u'_\nu(t))\, \mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_C} h(u_\nu(t))(\psi(\boldsymbol{v}_\tau) - \psi(\boldsymbol{u}'_\tau(t)))\, \mathrm{d}\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V}, \qquad (35)
\end{aligned}$$

where the function $\boldsymbol{f}\colon (0,T) \to V^*$ is defined by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_N(t), \boldsymbol{v})_{L^2(\Gamma_N;\mathbb{R}^d)} \qquad (36)$$

for all $\boldsymbol{v} \in V$ and a.e. $t \in (0,T)$.

We now combine inequality (35) with the constitutive law (19) and the initial conditions (24) to obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u}\colon Q \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon Q \to \mathbb{S}^d$ *such that*

$$\begin{aligned}
&\boldsymbol{\sigma}(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \\
&\quad + \int_0^t \mathcal{K}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}'(s)))\, ds \quad \text{for a.e. } t \in (0,T), \qquad (37)
\end{aligned}$$

$$\begin{aligned}
&\langle \boldsymbol{u}''(t), \boldsymbol{v} - \boldsymbol{u}'(t)\rangle_{V^*\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_C} j^0_\nu(t, u'_\nu(t); v_\nu - u'_\nu(t))\, \mathrm{d}\Gamma + \int_{\Gamma_C} h(u_\nu(t))(\psi(\boldsymbol{v}_\tau) - \psi(\boldsymbol{u}'_\tau(t)))\, \mathrm{d}\Gamma \\
&\quad \ge \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T), \qquad (38)
\end{aligned}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(0) = \boldsymbol{v}_0. \qquad (39)$$

Our main result in the study of Problem $\mathcal{P}_V$ is the following.

Theorem 8. *Suppose that (26)–(30), (31)*(a)-(d)*, (32) and (33) hold. If one of the following hypotheses*

$$i) \quad m_{\mathcal{A}} > 2\sqrt{2}(\sqrt{2}\overline{c}_{1\nu} + L_\psi \overline{h}\sqrt{|\Gamma_C|})\,\|\gamma\|^2 \tag{40}$$

$$ii) \quad (31)(\mathrm{e}) \text{ holds} \tag{41}$$

is satisfied, then Problem $\mathcal{P}_V$ has at least one solution which satisfies

$$\boldsymbol{u} \in W^{1,2}(0,T;V), \quad \boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V^*). \tag{42}$$

If, in addition, the following hypothesis holds

$$m_{\mathcal{A}} > m_\nu \|\gamma\|^2 + \sqrt{(L_{\mathcal{B}} + L_{\mathcal{K}})T} + L_h L\psi\, \|\gamma\| \sqrt{\|\gamma\| T}\,, \tag{43}$$

then the solution of Problem $\mathcal{P}_V$ is unique.

Proof. The proof consists of three main parts in which we establish existence, uniqueness and regularity of solution of Problem $\mathcal{P}_V$.

Existence part. To prove existence of solutions to Problem $\mathcal{P}_V$, we apply Theorem 6. To this end, let us denote $\boldsymbol{w}(t) = \boldsymbol{u}'(t)$, i.e., $\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0$ for a.e. $t \in (0,T)$. Using this notation and inserting (37) into (38), Problem $\mathcal{P}_V$ is equivalently formulated as follows.

Problem $\mathcal{P}_1$. *Find a velocity field $\boldsymbol{w}\colon Q \to \mathbb{R}^d$ such that*

$$\begin{aligned}
&\langle \boldsymbol{w}'(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^*\times V} + \big(\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{w}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{w}(t))\big)_{\mathcal{H}}\\
&\quad + \Big(\mathcal{B}\Big(t, \boldsymbol{\varepsilon}\big(\int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0\big)\Big) + \int_0^t \mathcal{K}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{w}(s)))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\Big)_{\mathcal{H}}\\
&\quad + \int_{\Gamma_C} j_\nu^0(t, w_\nu(t); v_\nu - w_\nu(t))\,\mathrm{d}\Gamma\\
&\quad + \int_{\Gamma_C} h\Big(\big(\int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0\big)_\nu\Big)\big(\psi(\boldsymbol{v}_\tau) - \psi(\boldsymbol{w}_\tau(t))\big)\,\mathrm{d}\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V}\\
&\qquad \text{for all } \ \boldsymbol{v} \in V, \ \text{a.e. } t \in (0,T),\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0.
\end{aligned}$$

We observe that if $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution to Problem $\mathcal{P}_V$, then $\boldsymbol{w} = \boldsymbol{u}'$ solves Problem $\mathcal{P}_1$. Vice versa, if $\boldsymbol{w}$ solves Problem $\mathcal{P}_1$, then we define $\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0$ for a.e. $t \in (0,T)$ and once we have displacement $\boldsymbol{u}$, the stress field in Problem $\mathcal{P}_V$ can be uniquely determined by (37), and then $(\boldsymbol{u}, \boldsymbol{\sigma})$ solves Problem $\mathcal{P}_V$. Therefore, in what follows, we will solve Problem $\mathcal{P}_1$ and we associate with it an inequality to which we apply Theorem 6.

We put $X = L^2(\Gamma_C;\mathbb{R}^d)$ and $Y = L^2(\Gamma_C)$, and introduce the following operators and functionals. Let $A\colon (0,T)\times V \to V^*$, $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}$, $\mathcal{R}\colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{R}_1\colon \mathcal{V} \to L^2(0,T;Y)$, $\varphi\colon Y \times X \to \mathbb{R}$, $J\colon (0,T)\times X \to \mathbb{R}$ and $M\colon V \to X$ be defined by

$$\langle A(t,\boldsymbol{u}), \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \text{for all } \ \boldsymbol{u}, \boldsymbol{v} \in V, \ \text{a.e. } t \in (0,T), \tag{44}$$

$$(\mathcal{S}\boldsymbol{w})(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0 \quad \text{for all } \ \boldsymbol{w} \in \mathcal{V}, \ \text{a.e. } t \in (0,T), \tag{45}$$

$$\langle(\mathcal{R}\boldsymbol{w})(t),\boldsymbol{v}\rangle_{V^*\times V}=\Big(\mathcal{B}\big(t,\boldsymbol{\varepsilon}((\mathcal{S}\boldsymbol{w})(t))\big)+\int_0^t\mathcal{K}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{w}(s)))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_{\mathcal{H}}$$
$$\text{for all } \boldsymbol{w}\in\mathcal{V},\ \boldsymbol{v}\in V,\ \text{a.e. } t\in(0,T), \tag{46}$$

$$(\mathcal{R}_1\boldsymbol{w})(t)=(\mathcal{S}\boldsymbol{w})_\nu(t)=\int_0^t w_\nu(s)\,ds+u_{0\nu}$$
$$\text{for all } \boldsymbol{w}\in\mathcal{V},\ \text{a.e. } t\in(0,T), \tag{47}$$

$$\varphi(\boldsymbol{y},\boldsymbol{z})=\int_{\Gamma_C}h(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}))\psi(\boldsymbol{x},\boldsymbol{z}_\tau(\boldsymbol{x}))\,\mathrm{d}\Gamma\quad\text{for all } \boldsymbol{y}\in Y,\boldsymbol{z}\in X, \tag{48}$$

$$J(t,\boldsymbol{z})=\int_{\Gamma_C}j_\nu(\boldsymbol{x},t,z_\nu(\boldsymbol{x}))\,\mathrm{d}\Gamma\quad\text{for all } \boldsymbol{z}\in X,\ \text{a.e. } t\in(0,T), \tag{49}$$

$$M=\gamma\colon V\to X \text{ is the trace operator.} \tag{50}$$

Using the above notation, we consider the following variational-hemivariational inequality.

Problem $\mathcal{P}_2$. *Find* $w\in\mathcal{W}$ *such that*

$$\begin{aligned}&\langle w'(t)+A(t,w(t))+(\mathcal{R}w)(t),v-w(t)\rangle_{V^*\times V}\\&+J^0(t,Mw(t);Mv-Mw(t))+\varphi((\mathcal{R}_1w)(t),Mv)-\varphi((\mathcal{R}_1w)(t),Mw(t))\\&\geq\langle f(t),v-w(t)\rangle_{V^*\times V}\quad\text{for all } v\in V,\ \text{a.e. } t\in(0,T),\\&\quad w(0)=v_0.\end{aligned}$$

In what follows we establish the existence of solutions to Problem $\mathcal{P}_2$ and we observe that every solution to Problem $\mathcal{P}_2$ is also a solution to Problem $\mathcal{P}_1$. We will verify hypotheses of Theorem 6.

1^0) From the proof of Theorem 14.2 of [33], we know that operator A defined by (44) satisfies hypothesis (1) with $m_A=m_{\mathcal{A}}$, $a_0(t)=\sqrt{2}\,\|\overline{a}_0(t)\|_{L^2(\Omega)}$, $a_1=\sqrt{2}\,\overline{a}_1$ and $\alpha_A=m_{\mathcal{A}}$.

2^0) The operator $M=\gamma\colon V\to X$ satisfies hypothesis (2). For the proof, we refer to Theorem 2.18 of [32].

3^0) Let $j\colon\Sigma_C\times\mathbb{R}^d\to\mathbb{R}$ be defined by

$$j(\boldsymbol{x},t,\boldsymbol{\xi})=j_\nu(\boldsymbol{x},t,\xi_\nu)\quad\text{for all } \boldsymbol{\xi}\in\mathbb{R}^d,\ \text{a.e. } (\boldsymbol{x},t)\in\Sigma_C.$$

Then, j satisfies the following properties.

I) The function $j(\cdot,\cdot,\boldsymbol{\xi})$ is measurable on Σ_C for all $\boldsymbol{\xi}\in\mathbb{R}^d$. Moreover, if $\mathbf{e}\in L^2(\Gamma_C)$ is as in (31)(a), then the function $\widetilde{e}(\boldsymbol{x})=\mathbf{e}(\boldsymbol{x})\,\boldsymbol{\nu}$ for a.e. $\boldsymbol{x}\in\Gamma_C$ satisfies $j(\cdot,\cdot,\widetilde{e}(\cdot))\in L^1(\Sigma_C)$.

II) The function $j(\boldsymbol{x},t,\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(\boldsymbol{x},t)\in\Sigma_C$.

III) By Proposition 3.37 of [30], we obtain

$$\partial j(\boldsymbol{x},t,\boldsymbol{\xi})\subset\partial j_\nu(\boldsymbol{x},t,\xi_\nu)\,\boldsymbol{\nu} \tag{51}$$

$$j^0(\boldsymbol{x},t,\boldsymbol{\xi};\boldsymbol{\eta})\leq j_\nu^0(\boldsymbol{x},t,\xi_\nu;\eta_\nu) \tag{52}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta}\in\mathbb{R}^d$, a.e. $(\boldsymbol{x},t)\in\Sigma_C$. From (31)(c) and inclusion (51), we deduce the following inequality

$$\|\partial j(\boldsymbol{x},t,\boldsymbol{\xi})\|\leq\overline{c}_{0\nu}(\boldsymbol{x},t)+\overline{c}_{1\nu}\|\boldsymbol{\xi}\|$$

for all $\boldsymbol{\xi}\in\mathbb{R}^d$, a.e. $(\boldsymbol{x},t)\in\Sigma_C$.

IV) Let $\boldsymbol{\zeta}_i \in \partial j(\boldsymbol{x}, t, \boldsymbol{\xi}_i)$, $\boldsymbol{\xi}_i$, $\boldsymbol{\zeta}_i \in \mathbb{R}^d$, $i = 1, 2$. Then, by hypothesis (31)(d) and inclusion (51), we have $\boldsymbol{\zeta}_i = \boldsymbol{\eta}_i \boldsymbol{\nu}$, $\boldsymbol{\eta}_i \in \partial j(\boldsymbol{x}, t, \xi_{i\nu})$ and

$$(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) = (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)\, \boldsymbol{\nu} \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)$$

$$= (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)(\xi_{1\nu} - \xi_{2\nu}) \geq -m_\nu \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$$

for a.e. $(\boldsymbol{x}, t) \in \Sigma_C$. This proves that

$$(\partial j(\boldsymbol{x}, t, \boldsymbol{\xi}_1) - \partial j(\boldsymbol{x}, t, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq -m_\nu \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$$

for all $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2 \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in \Sigma_C$.

V) From hypothesis (31)(e) and (52), we obtain

$$j^0(\boldsymbol{x}, t, \boldsymbol{\xi}; -\boldsymbol{\xi}) \leq j^0_\nu(\boldsymbol{x}, t, \xi_\nu; -\xi_\nu) \leq \overline{d}_\nu (1 + \|\boldsymbol{\xi}\|)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in \Sigma_C$.

Under the above notation, we consider the integral functional given by

$$J(t, \boldsymbol{z}) = \int_{\Gamma_C} j(\boldsymbol{x}, t, \boldsymbol{z}(\boldsymbol{x}))\, \mathrm{d}\Gamma \quad \text{for} \quad \boldsymbol{z} \in X, \ \text{a.e. } t \in (0, T).$$

Exploiting properties 3^0) I)-V), by Theorem 3.47 of [30], we have

(i) $J(\cdot, \boldsymbol{z})$ is measurable on $(0, T)$ for all $\boldsymbol{z} \in X$.

(ii) $J(t, \cdot)$ is well defined and Lipschitz on bounded subsets (hence also locally Lipschitz) on X for a.e. $t \in (0, T)$.

(iii) $\|\partial J(t, \boldsymbol{z})\|_{X^*} \leq c_0(t) + c_1 \|\boldsymbol{z}\|_X$ for all $\boldsymbol{z} \in X$, a.e. $t \in (0, T)$ with $c_0(t) = \sqrt{2|\Gamma_C|}\, \|\overline{c}_{0\nu}\|_{L^2(\Gamma_C)}$ and $c_1 = \sqrt{2}\, \overline{c}_{1\nu}$.

(iv) $\langle \partial J(t, \boldsymbol{z}_1) - \partial J(t, \boldsymbol{z}_2), \boldsymbol{z}_1 - \boldsymbol{z}_2 \rangle_{X^* \times X} \geq -m_J \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|^2_X$ for all $\boldsymbol{z}_i \in X$, $i = 1, 2$, a.e. $t \in (0, T)$ with $m_J = m_\nu$, cf. the proof of Theorem 5.23 of [30].

From conditions (i)-(iv) we concude that hypothesis (3) holds.

4^0) Next, we check that under hypotheses (29) and (30), the functional φ defined by (48) satisfies (4).

I) Let $\boldsymbol{z} \in X$ be fixed and define $g(\boldsymbol{x}, r) = \psi(\boldsymbol{x}, \boldsymbol{z}_\tau(\boldsymbol{x})) h(\boldsymbol{x}, r)$ for $r \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$. It is clear that $g(\cdot, r)$ is measurable for all $r \in \mathbb{R}$, $g(\boldsymbol{x}, \cdot)$ is continuous for a.e. $\boldsymbol{x} \in \Gamma_C$ and $|g(\boldsymbol{x}, r)| \leq \overline{h}\, \eta_z(\boldsymbol{x})$ for all $r \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$, where $\eta_z \in L^2(\Gamma_C)$ is given by

$$\eta_z(\boldsymbol{x}) = L_\psi \|\boldsymbol{z}_\tau(\boldsymbol{x})\| + L_\psi \|\boldsymbol{\xi}_0\| + |\psi(\boldsymbol{x}, \boldsymbol{\xi}_0)|$$

for a.e. $\boldsymbol{x} \in \Gamma_C$. From the Hölder inequality, we have

$$|\varphi(\boldsymbol{y}_1, \boldsymbol{z}) - \varphi(\boldsymbol{y}_2, \boldsymbol{z})| \leq L_h \int_{\Gamma_C} |\psi(\boldsymbol{x}, \boldsymbol{z}_\tau(\boldsymbol{x}))|\, |\boldsymbol{y}_1(\boldsymbol{x}) - \boldsymbol{y}_2(\boldsymbol{x})|\, \mathrm{d}\Gamma$$

$$\leq L_h \int_{\Gamma_C} |\eta_z(\boldsymbol{x})|\, |\boldsymbol{y}_1(\boldsymbol{x}) - \boldsymbol{y}_2(\boldsymbol{x})|\, \mathrm{d}\Gamma \leq L_h \|\eta_z\|_Y\, \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_Y$$

for all $\boldsymbol{y}_1$, $\boldsymbol{y}_2 \in Y$. Hence $\varphi(\cdot, \boldsymbol{z})$ is Lipschitz for all $\boldsymbol{z} \in X$ which implies that condition (4)(a) holds.

II) Let $\boldsymbol{y} \in Y$ be fixed and define $f(\boldsymbol{x}, \boldsymbol{\xi}) = h(\boldsymbol{x}, y(\boldsymbol{x})) \psi(\boldsymbol{x}, \boldsymbol{\xi}_\tau)$ for $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_C$. It is clear that $f(\cdot, \boldsymbol{\xi})$ is measurable for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $f(\boldsymbol{x}, \cdot)$ is Lipschitz continuous and convex for a.e. $\boldsymbol{x} \in \Gamma_C$. Using the Hölder inequality, we obtain

$$|\varphi(\boldsymbol{y}, \boldsymbol{z}_1) - \varphi(\boldsymbol{y}, \boldsymbol{z}_2)| \leq \int_{\Gamma_C} |h(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}))|\, |\psi(\boldsymbol{x}, \boldsymbol{z}_{1\tau}(\boldsymbol{x})) - \psi(\boldsymbol{x}, \boldsymbol{z}_{2\tau}(\boldsymbol{x}))|\, \mathrm{d}\Gamma$$

$$\leq L_\psi \overline{h} \int_{\Gamma_C} \|\boldsymbol{z}_{1\tau}(\boldsymbol{x}) - \boldsymbol{z}_{2\tau}(\boldsymbol{x})\| \, \mathrm{d}\Gamma \leq L_\psi \overline{h} \sqrt{|\Gamma_C|} \, \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_X.$$

Thus $\varphi(\boldsymbol{y}, \cdot)$ is convex and Lipschitz for all $\boldsymbol{y} \in Y$ which implies that condition (4)(b) is also satisfied. Moreover, by Remark 2, we deduce that

$$\|\partial\varphi(\boldsymbol{y}, \boldsymbol{z})\|_{X^*} \leq L_\psi \overline{h} \sqrt{|\Gamma_C|} \quad \text{for all } \boldsymbol{y} \in Y,\, \boldsymbol{z} \in X,$$

which implies condition (4)(c) with $c_\varphi = L_\psi \overline{h} \sqrt{|\Gamma_C|}$.
III) Finally, exploiting the Lipschitz continuity of $h(\boldsymbol{x}, \cdot)$ and $\psi(\boldsymbol{x}, \cdot)$ for a.e. $\boldsymbol{x} \in \Gamma_C$, by the Hölder inequality, we get

$$\begin{aligned}
&\varphi(\boldsymbol{y}_1, \boldsymbol{z}_2) - \varphi(\boldsymbol{y}_1, \boldsymbol{z}_1) + \varphi(\boldsymbol{y}_2, \boldsymbol{z}_1) - \varphi(\boldsymbol{y}_2, \boldsymbol{z}_2) \\
&\quad = \int_{\Gamma_C} (h(\boldsymbol{x}, \boldsymbol{y}_1(\boldsymbol{x})) - h(\boldsymbol{x}, \boldsymbol{y}_2(\boldsymbol{x})))\,(\psi(\boldsymbol{x}, \boldsymbol{z}_{2\tau}(\boldsymbol{x})) - \psi(\boldsymbol{x}, \boldsymbol{z}_{1\tau}(\boldsymbol{x})))\, \mathrm{d}\Gamma \\
&\quad \leq L_h L_\psi \int_{\Gamma_C} |\boldsymbol{y}_1(\boldsymbol{x}) - \boldsymbol{y}_2(\boldsymbol{x})| \, \|\boldsymbol{z}_{1\tau}(\boldsymbol{x}) - \boldsymbol{z}_{2\tau}(\boldsymbol{x})\| \, \mathrm{d}\Gamma \leq L_h L_\psi \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_Y \, \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_X.
\end{aligned}$$

Hence, condition (4)(d) is satisfied with $\beta_\varphi = L_h L_\psi$. This completes the proof that the functional φ satisfies hypothesis (4).
5^0) Since $c_1 = \sqrt{2}\, \overline{c}_{1\nu}$ by 3^0)V), $c_\varphi = L_\psi \overline{h} \sqrt{|\Gamma_C|}$ by 4^0)II), and $\alpha_A = m_{\mathcal{A}}$ by 1^0), it is clear that condition (40) implies hypothesis (5)(a).

The hypothesis (5)(b) is guaranteed by the condition (41). To see this, we recall that by 4^0)II), we have $\|\partial\varphi(\boldsymbol{y}, \boldsymbol{z})\|_{X^*} \leq L_\psi \overline{h} \sqrt{|\Gamma_C|}$ for all $\boldsymbol{y} \in Y$, $\boldsymbol{z} \in X$. Moreover, we make use of the inequality in 3^0)V) and formula

$$J^0(t, \boldsymbol{z}; \boldsymbol{w}) \leq \int_{\Gamma_C} j^0(\boldsymbol{x}, t, \boldsymbol{z}(\boldsymbol{x}); \boldsymbol{w}(\boldsymbol{x}))\, \mathrm{d}\Gamma \quad \text{for} \quad \boldsymbol{z}, \boldsymbol{w} \in X, \text{ a.e. } t \in (0, T), \tag{53}$$

which is a consequence of Theorem 3.47(iv) of [30] to conclude that

$$J^0(t, \boldsymbol{z}; -\boldsymbol{z}) \leq d_0\,(1 + \|\boldsymbol{z}\|_X) \quad \text{for all} \quad \boldsymbol{z} \in X, \text{ a.e. } t \in (0, T) \text{ with } d_0 \geq 0.$$

Hence (5)(b) is satisfied.
6^0) We will show that the operators $\mathcal{R}$ and $\mathcal{R}_1$ defined by (46) and (47) satisfy hypotheses (8)(a) and (b), respectively. Indeed, using (27)(b) and (28)(b), we obtain

$$\begin{aligned}
\|(\mathcal{R}\boldsymbol{w}_1)(t) - (\mathcal{R}\boldsymbol{w}_2)(t)\|_{V^*} &\leq \|\mathcal{B}(t, \varepsilon((\mathcal{S}\boldsymbol{w}_1)(t))) - \mathcal{B}(t, \varepsilon((\mathcal{S}\boldsymbol{w}_2)(t)))\|_{\mathcal{H}} \\
&\quad + \int_0^t \|\mathcal{K}(t-s, \varepsilon(\boldsymbol{w}_1(s))) - \mathcal{K}(t-s, \varepsilon(\boldsymbol{w}_2(s)))\|_{\mathcal{H}}\, ds \\
&\leq L_{\mathcal{B}}\, \|\varepsilon((\mathcal{S}\boldsymbol{w}_1)(t)) - \varepsilon((\mathcal{S}\boldsymbol{w}_2)(t))\|_{\mathcal{H}} + L_{\mathcal{K}} \int_0^t \|\varepsilon(\boldsymbol{w}_1(s)) - \varepsilon(\boldsymbol{w}_2(s))\|_{\mathcal{H}}\, ds \\
&\leq (L_{\mathcal{B}} + L_{\mathcal{K}}) \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V\, ds
\end{aligned} \tag{54}$$

for all $\boldsymbol{w}_1,\, \boldsymbol{w}_2 \in \mathcal{V}$, a.e. $t \in (0, T)$. It follows from (54) that

$$\|(\mathcal{R}\boldsymbol{w})(t)\|_{V^*} \leq (L_{\mathcal{B}} + L_{\mathcal{K}}) \|\boldsymbol{w}\|_{L^1(0,t;V)} + \|(\mathcal{R}\boldsymbol{0})(t)\|_{V^*} \tag{55}$$

for all $\boldsymbol{w} \in \mathcal{V}$, a.e. $t \in (0,T)$. Using (27)(c) and the definition of the operator $\mathcal{R}$, we infer that

$$\|(\mathcal{R}\mathbf{0})(t)\|_{V^*} \le \|\widetilde{b}(t)\|_{L^2(\Omega)} + \int_0^t \|\widetilde{k}(s)\|_{L^2(\Omega)}\, ds \tag{56}$$

with $\widetilde{b}(x,t) = \|\mathcal{B}(\boldsymbol{x},t,\mathbf{0})\|_{\mathbb{S}^d}$ and $\widetilde{k}(x,t) = \|\mathcal{K}(\boldsymbol{x},t,\mathbf{0})\|_{\mathbb{S}^d}$, $\widetilde{b}$, $\widetilde{k} \in L^2(Q)$. From (55) and (56), we deduce

$$\|\mathcal{R}\boldsymbol{w}\|_{\mathcal{V}^*} \le c(\|\boldsymbol{w}\|_{\mathcal{V}} + \|\widetilde{b}\|_{L^2(Q)} + \|\widetilde{k}\|_{L^2(Q)})$$

for all $\boldsymbol{w} \in \mathcal{V}$ with $c > 0$ which implies that operator $\mathcal{R}$ is well defined and has values in $\mathcal{V}^*$. Also, using the continuity of the trace operator and the inequality $|\xi_\nu| \le \|\boldsymbol{\xi}\|$ for $\boldsymbol{\xi} \in \mathbb{R}^d$, from the definition of operator $\mathcal{R}_1$, we get

$$\begin{aligned}
\|(\mathcal{R}_1\boldsymbol{w}_1)(t) - (\mathcal{R}_1\boldsymbol{w}_2)(t)\|_{V^*} &= \|(\mathcal{S}\boldsymbol{w}_1)_\nu(t) - (\mathcal{S}\boldsymbol{w}_2)_\nu(t)\|_Y \\
&\le \|\gamma\left((\mathcal{S}\boldsymbol{w}_1)(t) - (\mathcal{S}\boldsymbol{w}_2)(t)\right)\|_X \le \|\gamma\| \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V\, ds
\end{aligned} \tag{57}$$

for all $\boldsymbol{w}_1$, $\boldsymbol{w}_2 \in \mathcal{V}$, a.e. $t \in (0,T)$. We conclude that condition (8) is satisfied with $c_R = L_{\mathcal{B}} + L_{\mathcal{K}}$ and $c_{R_1} = \|\gamma\|$.
7^0) It is obvious that condition (6) is a consequence of the smallness condition (33). Finally, it is easy to see that (32) implies the regularity hypothesis (7).

All hypotheses of Theorem 6 are now verified, so we deduce from it that Problem $\mathcal{P}_2$ admits a unique solution $\boldsymbol{w} \in \mathcal{W}$. Next, using the inequality (53), we easily infer that the solution to Problem $\mathcal{P}_2$ is a solution to Problem $\mathcal{P}_1$. This implies that Problem $\mathcal{P}_V$ has at least one solution. The proof of the existence part of the theorem is complete.

Uniqueness part. In this part we assume, in addition to the previous hypotheses, that condition (43) holds. Since Problem $\mathcal{P}_V$ is equivalent to Problem $\mathcal{P}_1$, we show the uniqueness of Problem $\mathcal{P}_1$. Let $\boldsymbol{w}_1$, $\boldsymbol{w}_2 \in \mathcal{W}$ be solution to the following problem.

$$\begin{aligned}
&\langle \boldsymbol{w}'(t) + A(t,\boldsymbol{w}(t)) + (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^*\times V} \\
&\quad + \int_{\Gamma_C} j_\nu^0(t, w_\nu(t); v_\nu - w_\nu(t))\, \mathrm{d}\Gamma + \varphi((\mathcal{R}_1\boldsymbol{w})(t), \gamma\boldsymbol{v}) - \varphi((\mathcal{R}_1\boldsymbol{w})(t), \gamma\boldsymbol{w}(t)) \\
&\quad \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^*\times V} \quad \text{for all } \ \boldsymbol{v} \in V, \ \text{a.e. } t \in (0,T), \\
&\qquad \boldsymbol{w}(0) = \boldsymbol{v}_0.
\end{aligned}$$

We write the inequality satisfied by $\boldsymbol{w}_1$ and take $v = \boldsymbol{w}_2(t)$, and write the inequality with $\boldsymbol{w}_2$ and take $v = \boldsymbol{w}_1(t)$, and then we add the two resulting inequalities. We have

$$\begin{aligned}
&\langle \boldsymbol{w}_1'(t) - \boldsymbol{w}_2'(t), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V} \\
&\quad + \langle A(t,\boldsymbol{w}_1(t)) - A(t,\boldsymbol{w}_2(t)), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V} \\
&\quad \le \langle (\mathcal{R}\boldsymbol{w}_1)(t) - (\mathcal{R}\boldsymbol{w}_2)(t), \boldsymbol{w}_2(t) - \boldsymbol{w}_1(t)\rangle_{V^*\times V} \\
&\quad + \int_{\Gamma_C} \Big(j_\nu^0(t, w_{1\nu}(t); w_{2\nu}(t) - w_{1\nu}(t)) + j_\nu^0(t, w_{2\nu}(t); w_{1\nu}(t) - w_{2\nu}(t)) \Big)\, \mathrm{d}\Gamma \\
&\quad + \varphi((\mathcal{R}_1\boldsymbol{w}_1)(t), \gamma\boldsymbol{w}_2(t)) - \varphi((\mathcal{R}_1\boldsymbol{w}_1)(t), \gamma\boldsymbol{w}_1(t)) \\
&\quad + \varphi((\mathcal{R}_1\boldsymbol{w}_2)(t), \gamma\boldsymbol{w}_1(t)) - \varphi((\mathcal{R}_1\boldsymbol{w}_2)(t), \gamma\boldsymbol{w}_2(t)) \quad \text{a.e. } t \in (0,T), \\
&\quad \boldsymbol{w}_1(0) - \boldsymbol{w}_2(0) = 0.
\end{aligned}$$

Using Remark 5, (4)(d) and (31)(d), we obtain

$$\begin{aligned}
&\langle \boldsymbol{w}_1'(t) - \boldsymbol{w}_2'(t), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V} \\
&\quad + \langle A(t, \boldsymbol{w}_1(t)) - A(t, \boldsymbol{w}_2(t)), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V} \\
&\quad \le \langle (\mathcal{R}\boldsymbol{w}_1)(t) - (\mathcal{R}\boldsymbol{w}_2)(t), \boldsymbol{w}_2(t) - \boldsymbol{w}_1(t)\rangle_{V^*\times V} \\
&\quad + m_\nu \int\limits_{\Gamma_C} |w_{1\nu}(t) - w_{2\nu}(t)|^2 \,\mathrm{d}\Gamma \\
&\quad + \beta_\varphi \,\|(\mathcal{R}_1\boldsymbol{w}_1)(t) - (\mathcal{R}_1\boldsymbol{w}_2)(t)\|_Y \,\|\gamma\boldsymbol{w}_1(t) - \gamma\boldsymbol{w}_2(t)\|_X \quad \text{a.e. } t\in(0,T), \\
&\quad \boldsymbol{w}_1(0) - \boldsymbol{w}_2(0) = 0,
\end{aligned}$$

where $\beta_\varphi = L_h L_\psi$. Integrating on $(0,t)$, using (1)(e) and the initial condition, it follows

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_H^2 + m_{\mathcal{A}} \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V^2 \,ds \\
&\le \Big(\int_0^t \|(\mathcal{R}\boldsymbol{w}_1)(s) - (\mathcal{R}\boldsymbol{w}_2)(s)\|_{V^*}^2 \,ds\Big)^{1/2} \Big(\int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V^2 \,ds\Big)^{1/2} \\
&\quad + m_\nu \int_0^t \|\gamma w_1(s) - w_2(s)\|_X^2 \,ds \\
&\quad + \beta_\varphi \,\|\gamma\| \Big(\int_0^t \|(\mathcal{R}_1\boldsymbol{w}_1)(s) - (\mathcal{R}_1\boldsymbol{w}_2)(s)\|_Y^2 \,ds\Big)^{1/2} \Big(\int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V^2 \,ds\Big)^{1/2}
\end{aligned}$$

for all $t \in [0,T]$. Hence and from inequalities (54) and (57), we get

$$\begin{aligned}
m_{\mathcal{A}} \,\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(0,t;V)}^2 &\le \sqrt{c_R t}\, \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(0,t;V)}^2 \\
&\quad + m_\nu \|\gamma\|^2 \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(0,t;V)}^2 + \beta_\varphi \,\|\gamma\| \,\sqrt{c_{R_1} t}\, \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(0,t;V)}^2 \\
&\le \Big(m_\nu \|\gamma\|^2 + \sqrt{c_R T} + \beta_\varphi \|\gamma\| \sqrt{c_{R_1} T}\Big) \,\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(0,t;V)}^2
\end{aligned}$$

for all $t \in [0,T]$ with $\beta_\varphi = L_h L_\psi$, $c_R = L_{\mathcal{B}} + L_{\mathcal{K}}$ and $c_{R_1} = \|\gamma\|$. Hence and from hypothesis (43) it is clear that $\boldsymbol{w}_1 = \boldsymbol{w}_2$. This completes the proof of uniqueness of solution to Problem $\mathcal{P}_V$.

Regularity. In order to obtain the regularity (42) of the solution of Problem $\mathcal{P}_V$, we note that the regularity $\boldsymbol{w} \in \mathcal{W}$ together with the definition $\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + u_0$ for a.e. $t \in (0,T)$ combined with the continuity of the embedding $\mathcal{W} \subset C(0,T;H)$ implies that

$$\boldsymbol{u} \in W^{1,2}(0,T;V), \quad \boldsymbol{u}' \in C(0,T;H), \quad \boldsymbol{u}'' \in L^2(0,T;V^*). \tag{58}$$

In addition, the constitutive law (19), hypotheses (26), (27), (28) and (58) show that $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$. Furthermore, from (19) and (38) we obtain that $\boldsymbol{u}''(t) - \mathrm{Div}\,\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t)$ in Q. This equality combined with (32) and (58) imply that $\mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V^*)$. Thus, the regularity of the stress field is given by

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V^*). \tag{59}$$

Hence we conclude that the regularity (42) holds. This completes the proof of the theorem. □

We remark that the existence part of Theorem 8 represents a global result, and the uniqueness result is a local one since the length of the time interval has to satisfy the smallness condition (43).

A couple of functions $\boldsymbol{u}\colon (0,T) \to V$ and $\boldsymbol{\sigma}\colon (0,T) \to \mathcal{H}$ which satisfies (37), (38) and (39) is called a weak solution to Problem $\mathcal{P}$. In conclusion, we infer that under the assumptions of Theorem 8 there exists a unique weak solution to Problem $\mathcal{P}$ with regularity (42).

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Stanislaw Migórski and Justyna Ogorzaly
Faculty of Mathematics and Computer Science
Jagiellonian University in Krakow
ul. Łojasiewicza 6
30348 Krakow
Poland
e-mail: justyna.ogorzaly@gmail.com

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